UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical & Computer Engineering

ECE 204 *Numerical methods*

Approximating the integral using least-squares best-fitting polynomials

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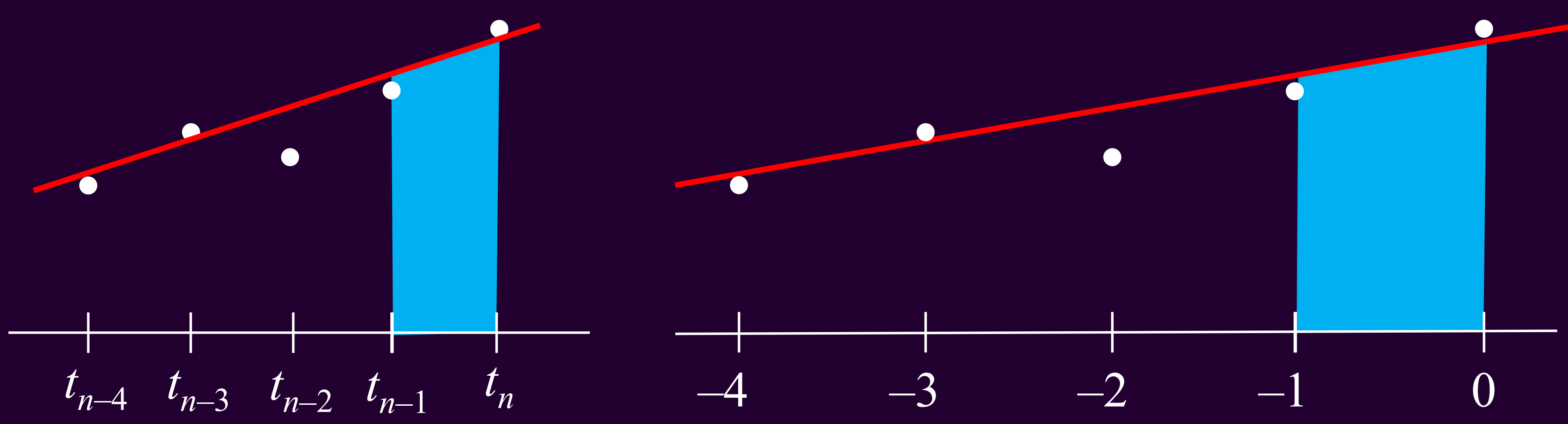
Introduction

- In this topic, we will
  - Discuss how to estimate an integral of data by using the least-squares best-fitting polynomials
  - Estimating $\int_{t_{n-1}}^{t_n} y(t)\,\mathrm{d}t$ or $\int_{t_n}^{t_{n+1}} y(t)\,\mathrm{d}t$ where $t_k = t_0 + kh$
  - Describe the formula for both linear and quadratic polynomials

Approximating the derivative

- In the last lecture, we saw that
  if we mapped the last $N + 1$ times $t_{n-N}, \ldots, t_n$, to $-N, \ldots, 0$,
  we can easily find $a_1\delta + a_0$ on the right
  - In this case, $y(t_n + \delta h) \approx a_0 + a_1\delta$
  - The scaling, however, increases the area by h

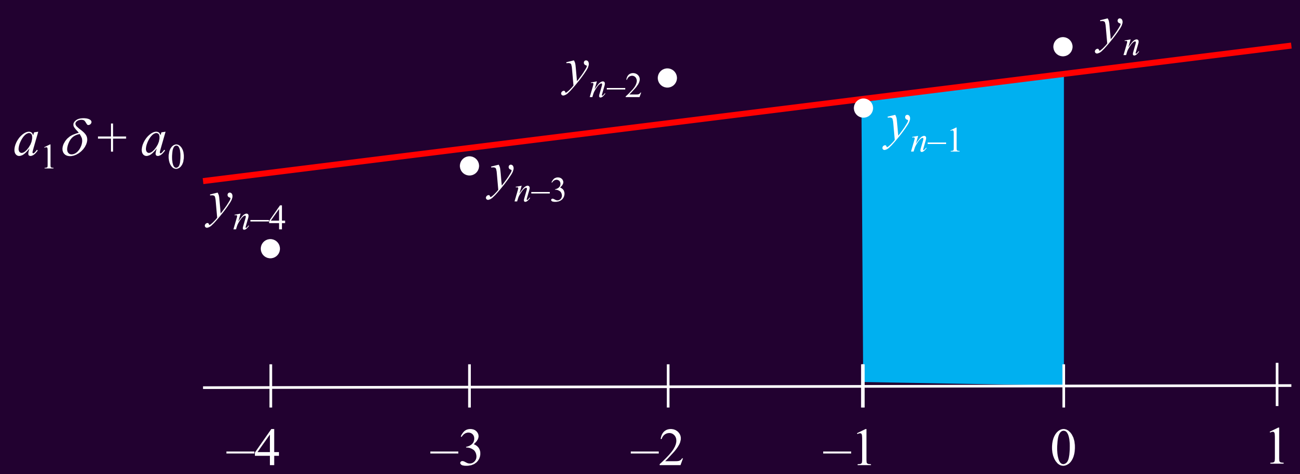
$$\delta \leftarrow \frac{t - t_n}{h}$$



Approximating the integral

- Thus, for a least-squares best-fitting linear polynomial, the best estimate of the integral

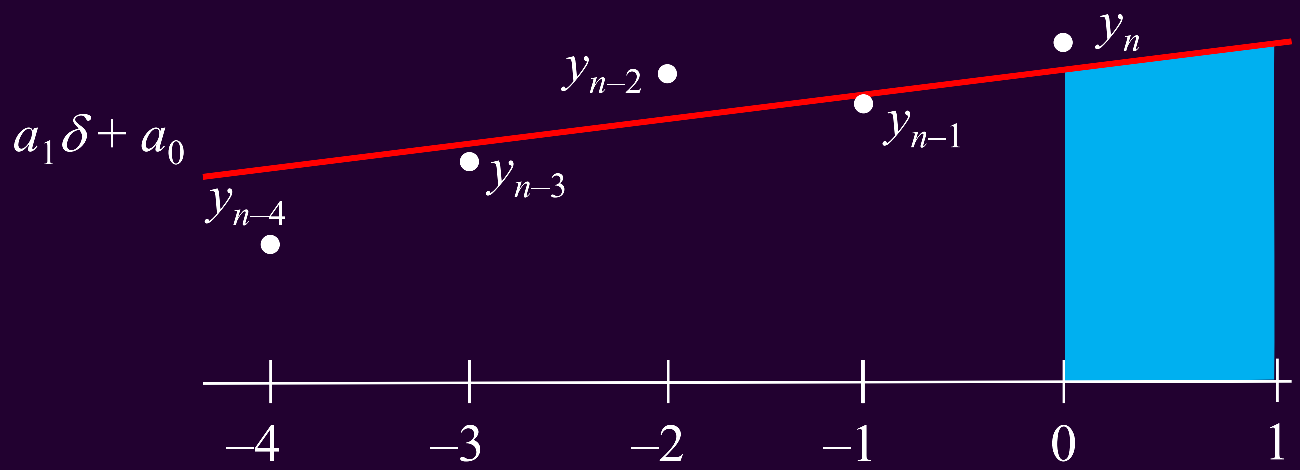
$$\int_{t_{n-1}}^{t_n} y(t)\,\mathrm{d}t \approx h\int_{-1}^{0} (a_1\delta + a_0)\,\mathrm{d}\delta$$
$$= h\left(a_0 - \frac{a_1}{2}\right)$$



Approximating the integral

- Thus, for a least-squares best-fitting linear polynomial, the best estimate of the integral

$$\int_{t_n}^{t_{n+1}} y(t)\,\mathrm{d}t \approx h\int_0^1 (a_1\delta + a_0)\,\mathrm{d}\delta$$
$$= h\left(a_0 + \frac{a_1}{2}\right)$$

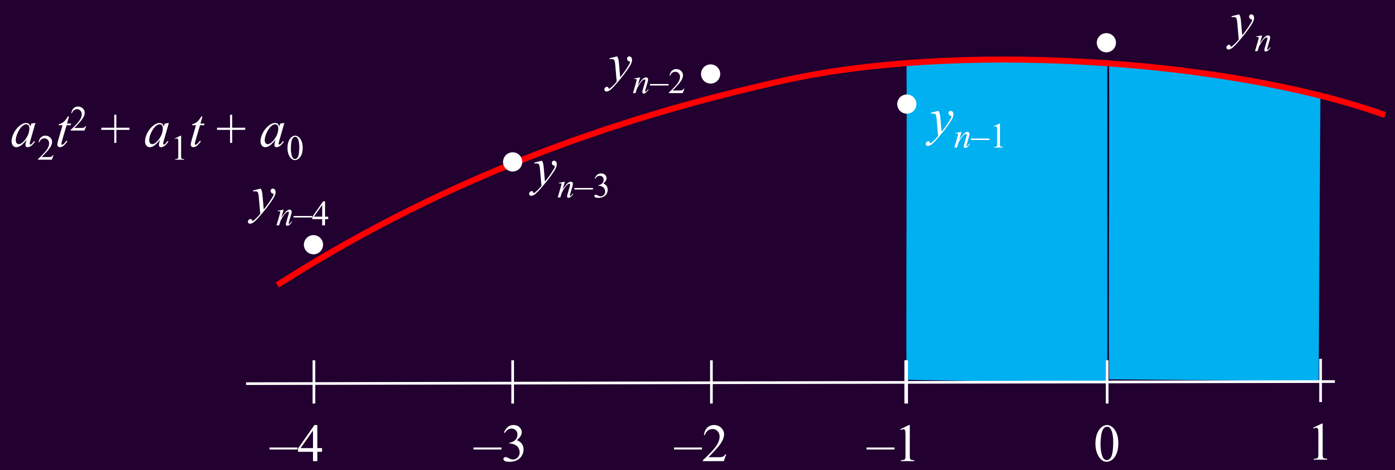


Approximating the integral

- We can perform the same operation for a least-squares quadratic polynomial

$$\int_{t_{n-1}}^{t_n} y(t)\,\mathrm{d}t \approx h\int_{-1}^{0}\left(a_2\delta^2 + a_1\delta + a_0\right)\mathrm{d}\delta = h\left(a_0 - \frac{a_1}{2} + \frac{a_2}{3}\right)$$

$$\int_{t_n}^{t_{n+1}} y(t)\,\mathrm{d}t \approx h\int_{0}^{1}\left(a_2\delta^2 + a_1\delta + a_0\right)\mathrm{d}\delta = h\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3}\right)$$



O(1) run time

This exemplifies an idea,
it is not required for this course.

- Everything in this class runs in O(1) time with O(n) memory:

```
class Estimate {
    public:
        Estimate( double y0, double delta_t );
        double operator()( double delta ) const;
        void next( double y );
        double diff() const;
        double int_next() const;
        double int_prev() const;
private:
        std::size_t curr_;
        double      ys_[4];
        double      a1_, a0_, s_;
        double      delta_t_;
};
```

```
// Estimate integral over the previous time step
double Estimate::int_prev() const {
    return delta_t_*(a0_ - a1_/2.0);
}

// Extrapolate integral over the next time step
double Estimate::int_prev() const {
    return delta_t_*(a0_ + a1_/2.0);
}
```

Summary

- Following this topic, you now
  - Understand how to estimate the integral using least-squares best-fitting polynomials
  - Are aware that we can both estimate the integral over the last time interval, or extrapolate and estimate the integral over the next time interval
  - Understand that if we already have the coefficients, we can find these estimates in O(1) time

References

[1] https://en.wikipedia.org/wiki/Least_squares

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.

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